

Math 7581 project

Area preserving dynamics of a closed curve in 2D

Sumit Jain and Karthik Raveendran
sumit@cc.gatech.edu, kraveendran@gatech.edu
College of Computing
Georgia Institute of Technology

Contents

1	Introduction	2
2	Problem formulation	2
3	Derivation of equations of motion	3
3.1	Euler-Lagrange equation in multiple variables	3
3.2	Application to the surface tension problem	4
3.3	Different potential energy	4
4	Discretization of the curve	5
4.1	Kinetic energy	5
4.2	Potential energy	6
4.3	Constraints	6
4.3.1	Constant area	6
4.3.2	Constant mass	7
4.4	Simplification	7
4.5	Euler-Lagrange equations	8
5	Numerical integration	8
5.1	Variational integrator	9
6	Results	10
A	Vector product variant of integration by parts	13
B	Vector product variant of the fundamental lemma of calculus of variations	13

1 Introduction

Surface tension provides necessary forces for the formation of bubbles with water. The tendency to minimize this wall tension causes the bubble to evolve into a spherical shape with complex dynamics that are visually appealing. We propose a variational formulation for the dynamics of an incompressible bubble under the influence of surface tension forces. We derive the Euler-Lagrange equations for a continuous closed curve in 2D and discuss the discretization of this problem and approximations that allow us to simulate this problem efficiently on a computer.

2 Problem formulation

Given a continuous closed curve in 2D, our goal is to preserve the area enclosed by the curve while minimizing its length. We represent the boundary of the curve parametrically by its arc length θ at any time t as $\mathbf{x}(\theta, t) = (x(\theta, t), y(\theta, t))^T$ ($\theta \in [0, L]$, L is the length of the curve) with the condition $\mathbf{x}(0, t) = \mathbf{x}(L, t)$ for the loop to be closed. i.e. $L = \int_0^L \left| \frac{\partial \mathbf{x}}{\partial \theta} \right| d\theta$ and $\left| \frac{\partial \mathbf{x}}{\partial \theta} \right| = 1$. Note that the length L is a function of time $L(t)$. Now we make a change of coordinates such that the new parameter s represents a scaled length i.e. $s = \frac{\theta}{L}$. Therefore $s \in [0, 1]$ spans the curve for any time t . Now, we can compute $\frac{\partial \mathbf{x}}{\partial s}$ as:

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial s} &= \frac{\partial \mathbf{x}}{\partial \theta} \frac{\partial \theta}{\partial s} \\ &= L \frac{\partial \mathbf{x}}{\partial \theta} \end{aligned} \quad (1)$$

Therefore $\left| \frac{\partial \mathbf{x}}{\partial s} \right| = L(t)$ and is thus only a function of time. In all the further discussion, we use this scaled length parameter s for representing spatial quantities.

We would like to conserve the mass of the curve as well as its area as it evolves in time. These constraints are enforced through the use of Lagrange multipliers. It is useful to explicitly define all the quantities. Partial derivatives with respect to s and t will be represented by prime and dot respectively. At any time t , we define the potential energy as scaled length of the curve:

$$PE = k \int |\mathbf{x}'| ds$$

We assume uniform density ρ for the curve at any given time t . Hence the kinetic energy can be derived by integrating along the curve. Lets consider infinitesimal element on the curve. The mass dm is given by $\rho(t)|\mathbf{x}'|ds$. The square of velocity is given by $\dot{\mathbf{x}}^2 = \dot{x}^2 + \dot{y}^2$. Now, the kinetic energy is computed as:

$$KE = \int \rho(t) \frac{1}{2} |\mathbf{x}'| \dot{\mathbf{x}}^2 ds$$

To preserve the area enclosed by the curve \mathbf{x} , we have the area constraint:

$$\begin{aligned} \int y x' ds &= A_0 \\ \Rightarrow \int \mathbf{x}^T A \mathbf{x}' ds &= A_0 \end{aligned}$$

where $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Since the length of the curve can change, we need to enforce that the total mass remains constant:

$$\begin{aligned} \int \rho(t)|\mathbf{x}'|ds &= m \\ \rho(t)|\mathbf{x}'| &= m \end{aligned} \quad (2)$$

The second step is true since both the quantities inside the integral are independent of s .

Now we can write the functional as $\int (KE - PE - \lambda_1(\text{Area constraint}) - \lambda_2(\text{Mass constraint}))dt$ and the Lagrangian as:

$$\begin{aligned} F(\mathbf{x}, \mathbf{x}', \dot{\mathbf{x}}) &= \frac{1}{2}\rho(t)|\mathbf{x}'|\dot{\mathbf{x}}^2 - k|\mathbf{x}'| - \lambda_1\mathbf{x}^T A\mathbf{x}' - \lambda_2\rho(t)|\mathbf{x}'| \\ &= \frac{1}{2}m\dot{\mathbf{x}}^2 - k|\mathbf{x}'| - \lambda_1\mathbf{x}^T A\mathbf{x}' \end{aligned} \quad (3)$$

Note that we ignored the term for mass constraint since we replaced $\rho(t)|\mathbf{x}'|$ by m which is independent both of s and t .

3 Derivation of equations of motion

3.1 Euler-Lagrange equation in multiple variables

In this section, we derive the Euler-Lagrange equations for a general functional form in two variables. In our case, for a two variable vector function $\mathbf{x}(s, t)$, the functional \mathcal{F} is defined as:

$$\mathcal{F}[\mathbf{x}] = \iint F(s, t, \mathbf{x}, \mathbf{x}', \dot{\mathbf{x}}) ds dt \quad (4)$$

To derive the Euler-Lagrange equations in two variables, we note that for the functional \mathcal{F} to have an extreme value, its variation must vanish at that point. We can compute the first variation by considering any smooth vector test function $\mathbf{v}(s, t)$ inside the domain D , such that it vanishes on the boundary. Then, the first variation of the functional $\delta\mathcal{F}_{\mathbf{x}}[\mathbf{v}]$ can be written as:

$$\delta\mathcal{F}_{\mathbf{x}}[\mathbf{v}] = \iint (F_{\mathbf{x}}\mathbf{v} + F_{\mathbf{x}'}\mathbf{v}' + F_{\dot{\mathbf{x}}}\dot{\mathbf{v}}) ds dt = 0 \quad (5)$$

where $F_{\mathbf{z}}$ denotes the partial derivative $\frac{\partial F}{\partial \mathbf{z}}$ for a vector quantity \mathbf{z} .

Note that:

$$\iint (F_{\mathbf{x}'}\mathbf{v}' + F_{\dot{\mathbf{x}}}\dot{\mathbf{v}}) ds dt = \iint \left(\frac{\partial(F_{\mathbf{x}'}\mathbf{v})}{\partial s} + \frac{\partial(F_{\dot{\mathbf{x}}}\mathbf{v})}{\partial t} \right) ds dt - \iint \left(\frac{\partial}{\partial s}F_{\mathbf{x}'} + \frac{\partial}{\partial t}F_{\dot{\mathbf{x}}} \right) \mathbf{v} ds dt \quad (6)$$

$$= \int_{\Gamma} (F_{\mathbf{x}'}dt - F_{\dot{\mathbf{x}}}ds)\mathbf{v} - \iint \left(\frac{\partial}{\partial s}F_{\mathbf{x}'} + \frac{\partial}{\partial t}F_{\dot{\mathbf{x}}} \right) \mathbf{v} ds dt \quad (7)$$

We apply Green's theorem to the first term of Equation (6) to obtain the first term of Equation (7). However, since \mathbf{v} is zero along the boundary, this term becomes zero. Hence we now have:

$$\iint \left(F_{\mathbf{x}} - \frac{\partial}{\partial s} F_{\mathbf{x}'} - \frac{\partial}{\partial t} F_{\dot{\mathbf{x}}} \right) \mathbf{v} ds dt = 0 \quad (8)$$

Applying the vector product variant of the fundamental lemma of the calculus of variations (Appendix B), we get the Euler-Lagrange equations:

$$F_{\mathbf{x}} - \frac{\partial}{\partial s} F_{\mathbf{x}'} - \frac{\partial}{\partial t} F_{\dot{\mathbf{x}}} = \mathbf{0}^T \quad (9)$$

This vector equation represents a coupled system of partial differential equations (PDEs).

3.2 Application to the surface tension problem

For our problem, we have the Lagrangian (including the Lagrange multipliers) defined as:

$$F(\mathbf{x}, \mathbf{x}', \dot{\mathbf{x}}) = \frac{1}{2} m \dot{\mathbf{x}}^2 - k |\mathbf{x}'| - \lambda_1 \mathbf{x}^T A \mathbf{x}' \quad (10)$$

We write out each term of the Euler-Lagrange equation separately for clarity:

$$F_{\mathbf{x}}^T = -\lambda_1 A \mathbf{x}' \quad (11)$$

$$\frac{\partial}{\partial s} F_{\mathbf{x}'}^T = -\frac{k}{|\mathbf{x}'|} \mathbf{x}'' - \lambda_1 A^T \mathbf{x}' \quad (12)$$

$$\frac{\partial}{\partial t} F_{\dot{\mathbf{x}}}^T = m \ddot{\mathbf{x}} \quad (13)$$

Therefore, the E-L equations are a system of non-linear PDEs.

3.3 Different potential energy

We can choose a different potential energy formulation to mimic a similar behavior as surface tension as long as the curve for minimum energy is a circle. Therefore, we define a new potential energy as:

$$PE = k \int \mathbf{x}'^2 ds \quad (14)$$

This can be thought of as a “spring” energy for each infinitesimal element of the curve. We will see in the next section that this new definition leads to much simpler expression for the discrete case of Euler-Lagrange equations.

Now, the Lagrangian can be written as:

$$F(\mathbf{x}, \mathbf{x}', \dot{\mathbf{x}}) = \frac{1}{2} m \dot{\mathbf{x}}^2 - k \mathbf{x}'^T \mathbf{x}' - \lambda_1 \mathbf{x}^T A \mathbf{x}' \quad (15)$$

Again, we write out each term of the Euler-Lagrange equation:

$$F_{\mathbf{x}}^T = -\lambda_1 A \mathbf{x}' \quad (16)$$

$$\frac{\partial}{\partial s} F_{\mathbf{x}'}^T = -k \mathbf{x}'' - \lambda_1 A^T \mathbf{x}' \quad (17)$$

$$\frac{\partial}{\partial t} F_{\dot{\mathbf{x}}}^T = m \ddot{\mathbf{x}} \quad (18)$$

In the next section, we derive the discrete version of the Euler-Lagrange equations by first approximating the curve by finite number of linear segments and then discretizing in time to numerically evolve the discrete curve in time.

4 Discretization of the curve

We adopt a numerical approach to evolve the state of the curve in time. The first step is to approximate the curve by discretizing the boundary. We linearly approximate the boundary by finite number of vertices and edges. Each vertex is shared by two edges forming a poly-loop. We denote the number of vertices or edges by n . Following directly from the parametric representation, the position of i^{th} vertex is given by $\mathbf{x}_i = (x_i, y_i)^T$. We stack all these positions in a single vector \mathbf{x} representing the position of the curve. Hence, $\mathbf{x} = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)^T$. Similarly the velocity is represented by $\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_n)^T$.

We now write the Lagrangian for the system and derive the discrete Euler-Lagrange equations governing the time evolution of this poly-loop.

4.1 Kinetic energy

Assume a uniform density ρ for the poly-loop. We now compute the kinetic energy of the system by summing over the kinetic energies of each edge. The end points of this edge are vertices \mathbf{x}_i and \mathbf{x}_{i+1} . We represent any point on this edge parametrically as $\mathbf{x}(s) = \mathbf{x}_i + ns(\mathbf{x}_{i+1} - \mathbf{x}_i)$ (where $s \in [0, \frac{1}{n}]$) or $\mathbf{x}(s) = \mathbf{x}_i + n\Delta\mathbf{x}_i s$. Now we integrate over this edge to find the kinetic energy. The differential mass of each segment is given by:

$$\begin{aligned} dm &= \rho \sqrt{x'^2 + y'^2} ds \\ &= \rho n |\Delta\mathbf{x}_i| ds \end{aligned} \quad (19)$$

Since the edge is linear, x' and y' are constants and are equal to $n\Delta x_i = n(x_{i+1} - x_i)$ and $n\Delta y_i = n(y_{i+1} - y_i)$ respectively. The kinetic energy of the edge can be written as:

$$\begin{aligned} KE_i &= \int_0^{\frac{1}{n}} \frac{1}{2} dm \dot{\mathbf{x}}^2(s) \\ &= \frac{1}{2} \rho n |\Delta\mathbf{x}_i| \int_0^{\frac{1}{n}} (\dot{\mathbf{x}}_i + n\Delta\dot{\mathbf{x}}_i s)^2 ds \\ &= \frac{1}{2} \rho n |\Delta\mathbf{x}_i| \left(\frac{\dot{\mathbf{x}}_i^2 + \dot{\mathbf{x}}_i^T \Delta\dot{\mathbf{x}}_i + \frac{\Delta\dot{\mathbf{x}}_i^2}{3}}{n} \right) \\ &= \frac{1}{6} \rho |\Delta\mathbf{x}_i| (\dot{\mathbf{x}}_i^2 + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_{i+1} + \dot{\mathbf{x}}_{i+1}^2) \end{aligned}$$

The total kinetic energy of the system is computed by summing up kinetic energies for each edge:

$$KE = \sum_{i=1}^n \frac{1}{6} \rho |\Delta \mathbf{x}_i| (\dot{\mathbf{x}}_i^2 + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_{i+1} + \dot{\mathbf{x}}_{i+1}^2) \quad (20)$$

4.2 Potential energy

We define the potential energy such that the system tries to go towards a circular configuration. For example, surface tension tries to minimize the length of the curve. We defined a variant of this energy in the continuous case as $k \int (x'(s)^2 + y'(s)^2) ds$ or $k \int \mathbf{x}'(s)^2 ds$. We now show that the discrete version of this potential energy leads to a simple expression which can be expressed in a bi-linear form. Potential energy for each edge is given by:

$$\begin{aligned} PE_i &= k \int_0^{\frac{1}{n}} (x'(s)^2 + y'(s)^2) ds \\ &= kn \Delta \mathbf{x}_i^2 \\ &= kn (\dot{\mathbf{x}}_i^2 - 2\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_{i+1} + \dot{\mathbf{x}}_{i+1}^2) \end{aligned} \quad (21)$$

Thus the total energy can be represented as:

$$\begin{aligned} PE &= \sum_{i=1}^n kn (\dot{\mathbf{x}}_i^2 - 2\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_{i+1} + \dot{\mathbf{x}}_{i+1}^2) \\ &= \mathbf{x}^T U \mathbf{x} \end{aligned} \quad (22)$$

where U is given by:

$$U = kn \begin{pmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix} \quad (23)$$

$$U_1 = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & -1 & 2 & -1 \\ -1 & 0 & \dots & -1 & 2 \end{pmatrix} \quad (24)$$

4.3 Constraints

4.3.1 Constant area

To compute the area enclosed by the loop, we sum up the signed areas under each edge. The signed area under each edge is given by:

$$A_i = \frac{1}{2} (x_{i+1} - x_i)(y_{i+1} + y_i) \quad (25)$$

The total area is given by the bi-linear form:

$$\begin{aligned} \text{Area} &= \sum_{i=1}^n \frac{1}{2} (x_{i+1} - x_i)(y_{i+1} + y_i) \\ &= \mathbf{x}^T A \mathbf{x} \end{aligned} \quad (26)$$

where the matrix A is given by:

$$A = \frac{1}{4} \begin{pmatrix} \mathbf{0} & A_1 \\ A_1^T & \mathbf{0} \end{pmatrix} \quad (27)$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & -1 \\ -1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & -1 & 0 & 1 \\ 1 & 0 & \dots & -1 & 0 \end{pmatrix} \quad (28)$$

Now, the area constraint can be written as $\mathbf{x}^T A \mathbf{x} = A_0$ for some constant area A_0 .

4.3.2 Constant mass

To conserve mass, we sum the mass over the entire loop and get the mass constraint as $\rho \sum |\Delta \mathbf{x}_i| = M_0$ for some constant mass M_0 .

4.4 Simplification

The Lagrangian for this system can be written as:

$$\begin{aligned} L(\mathbf{x}, \dot{\mathbf{x}}) &= KE - PE - \lambda_1(\text{area constraint}) - \lambda_2(\text{mass constraint}) \\ L(\mathbf{x}, \dot{\mathbf{x}}) &= \sum_{i=1}^n \frac{1}{6} \rho |\Delta \mathbf{x}_i| (\dot{\mathbf{x}}_i^2 + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_{i+1} + \dot{\mathbf{x}}_{i+1}^2) - \mathbf{x}^T U \mathbf{x} \\ &\quad - \lambda_1(\mathbf{x}^T A \mathbf{x} - A_0) - \lambda_2(\rho \sum_{i=1}^n |\Delta \mathbf{x}_i| - M_0) \end{aligned} \quad (29)$$

We notice that the first and the last terms are very non-linear because of the lengths of the edges involved. We make a simplification so that the Lagrangian becomes simple. If we fix the mass of each edge, the density will not be uniform, but the total mass will always be conserved. i.e. $\rho |\Delta \mathbf{x}_i|$ is constant ($= m_i$) and thus the total mass $\sum \rho |\Delta \mathbf{x}_i|$ automatically becomes constant. The kinetic energy now becomes:

$$\begin{aligned} KE &= \sum_{i=1}^n \frac{1}{6} m_i (\dot{\mathbf{x}}_i^2 + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_{i+1} + \dot{\mathbf{x}}_{i+1}^2) \\ &= \dot{\mathbf{x}}^T M \dot{\mathbf{x}} \end{aligned} \quad (30)$$

where M is given by:

$$M = \frac{1}{6} \begin{pmatrix} M_1 & \mathbf{0} \\ \mathbf{0} & M_1 \end{pmatrix} \quad (31)$$

$$M_1 = \begin{pmatrix} m_1 + m_2 & 0.5m_1 & 0 & \dots & 0.5m_n \\ 0.5m_1 & m_2 + m_3 & 0.5m_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0.5m_{n-2} & m_{n-1} + m_n & 0.5m_{n-1} \\ 0.5m_n & 0 & \dots & 0.5m_{n-1} & m_n + m_1 \end{pmatrix} \quad (32)$$

Since the total mass is implicitly constrained, we do not need the second constraint for mass anymore.

4.5 Euler-Lagrange equations

Now, we derive the Euler-Lagrange equations for the discrete case with our energy and constraints formulations. The functional for the state $\mathbf{x}(t)$ can be written as:

$$\mathcal{F}[\mathbf{x}] = \int_0^T (\dot{\mathbf{x}}^T M \dot{\mathbf{x}} - \mathbf{x}^T U \mathbf{x} - \lambda(\mathbf{x}^T A \mathbf{x} - A_0)) dt \quad (33)$$

Note that the constraint term occurs inside the integral since it acts as “constraint force” which is time dependent and acts instantaneously to correct the area along some direction which depends on \mathbf{x} .

We now compute the first variation of this functional and set it to zero to get the Euler-Lagrange equations. We vary along a direction \mathbf{v} which is zero on the boundary:

$$\begin{aligned} \delta \mathcal{F}_{\mathbf{x}}[\mathbf{v}] &= \lim_{\tau \rightarrow 0} \frac{d}{d\tau} \int_0^T ((\dot{\mathbf{x}} + \tau \dot{\mathbf{v}})^T M (\dot{\mathbf{x}} + \tau \dot{\mathbf{v}}) - (\mathbf{x} + \tau \mathbf{v})^T U (\mathbf{x} + \tau \mathbf{v}) - \lambda((\mathbf{x} + \tau \mathbf{v})^T A (\mathbf{x} + \tau \mathbf{v}) - A_0)) dt \\ \delta \mathcal{F}_{\mathbf{x}}[\mathbf{v}] &= 2 \int_0^T (\dot{\mathbf{x}}^T M \dot{\mathbf{v}} - \mathbf{x}^T U \mathbf{v} - \lambda \mathbf{x}^T A \mathbf{v}) dt \end{aligned} \quad (34)$$

Applying integration by parts on the first term with $\dot{\mathbf{v}}$ (see Appendix A), we get:

$$\delta \mathcal{F}_{\mathbf{x}}[\mathbf{v}] = \dot{\mathbf{x}}^T M \mathbf{v} \Big|_0^T - 2 \int_0^T \ddot{\mathbf{x}}^T M \mathbf{v} dt - 2 \int_0^T \mathbf{x}^T (U + \lambda A) \mathbf{v} dt \quad (35)$$

$$= 0 - 2 \int_0^T (\ddot{\mathbf{x}}^T M + \mathbf{x}^T (U + \lambda A)) \mathbf{v} dt \quad (36)$$

$$\Rightarrow \int_0^T (M \ddot{\mathbf{x}} + (U + \lambda A) \mathbf{x})^T \mathbf{v} dt = 0 \quad (37)$$

This is true for all \mathbf{v} in the set of admissible variations, hence we can apply a vector product variant of the fundamental lemma of calculus of variations (see Appendix B).

Therefore the Euler-Lagrange equations are:

$$M \ddot{\mathbf{x}} + (U + \lambda A) \mathbf{x} = 0 \quad (38)$$

with the constraint $\mathbf{x}^T A \mathbf{x} = A_0$.

5 Numerical integration

The Euler-Lagrange equations governing the dynamics of the system are a system of first order ODEs. In general, it is easy to solve a first order system, but we have constraint to satisfy as well. Therefore, solving for λ becomes non-trivial. To this end, we discretize our dynamics in

time as well and thus numerically integrate Equation 38. There can be many different ways to discretize an equation for numerical integration. But we attempt to discretize in a way such that the physical properties of the system such as momentum and energy are conserved. There is a class of numerical integrators known as “variational” integrators; these integration schemes can be derived in a way such that they preserve desired quantities. For our problem, we derive a first-order accurate integration scheme which turns out to be very simple as well as preserves energy (i.e. there is no numerical drift).

5.1 Variational integrator

As described in [Stern and Desbrun 2006], we discretize the Lagrangian and then derive the discrete Euler-Lagrange equations rather than directly discretizing Equation 38.

We replace the continuous function $\mathbf{x}(t)$ by a discrete set of values $(\mathbf{x}_1, \mathbf{x}_1, \dots, \mathbf{x}_n)$ corresponding to times $(t_1 = 0, t_1, \dots, t_n = T)$. We now approximate the Lagrangian L on a time interval $[t_k, t_{k+1}]$ by a discrete Lagrangian $L_d(\mathbf{x}_k, \mathbf{x}_{k+1})$ as:

$$L_d(\mathbf{x}_k, \mathbf{x}_{k+1}) \approx \int_{t_k}^{t_{k+1}} L(\mathbf{x}, \dot{\mathbf{x}}) dt \quad (39)$$

We approximate the right hand side by a one point quadrature i.e interval length times the value of the integrand at one point, and the velocity by $\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{h}$ (where h is the chosen constant step size):

$$L_d(\mathbf{x}_k, \mathbf{x}_{k+1}) = hL\left(\mathbf{x}_k, \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{h}\right) \quad (40)$$

We then compute the first variation in the discrete setting in a fashion very similar to the continuous case and set it to zero (For details of derivation, see [Stern and Desbrun 2006]). The discrete Euler-Lagrange equations are then given by:

$$D_1 L_d(\mathbf{x}_k, \mathbf{x}_{k+1}) + D_2 L_d(\mathbf{x}_{k-1}, \mathbf{x}_k) = 0 \quad (41)$$

where D_1 and D_2 are the partial differential operators with respect to first and second arguments respectively.

Applying Equation 41 to our problem, we get the discrete Euler-Lagrange equations as:

$$M \frac{\mathbf{x}_{k+1} - 2\mathbf{x}_k + \mathbf{x}_{k-1}}{h^2} + (U + \lambda A)\mathbf{x}_k = 0 \quad (42)$$

with the constraint $\mathbf{x}_{k+1}^T A \mathbf{x}_{k+1} = 0$. Therefore, given \mathbf{x}_{k-1} and \mathbf{x}_k , we can compute \mathbf{x}_{k+1} in terms of λ :

$$\begin{aligned} \mathbf{x}_{k+1} &= 2\mathbf{x}_k - \mathbf{x}_{k-1} - h^2 M^{-1}(U + \lambda A)\mathbf{x}_k \\ &= (2\mathbf{x}_k - \mathbf{x}_{k-1} - h^2 M^{-1}U\mathbf{x}_k) - \lambda(h^2 M^{-1}A\mathbf{x}_k) \end{aligned} \quad (43)$$

We substitute this value of \mathbf{x}_{k+1} in the constraint. We get a quadratic equation in λ which can be solved easily (we choose the value which has the smaller magnitude; the other value which is much larger flips the curve). We then substitute this value back in Equation 43 and get the state \mathbf{x}_{k+1} at time t_{k+1} .

6 Results

In order to test our numerical integrator, we designed a simple interface that allows us to sketch out curves by hand as well as load up prescribed analytic shapes. First, we look at the energy preserving dynamics of an undamped ellipse ($a = 2$, $b = 0.4$). We pick $k = 1000$ and simulate the dynamics for over 3 seconds of real world time (corresponding to over 2000 timesteps). We plot the kinetic, potential and total energy over real world time in Figure 1. In this example, the potential energy drops to its lowest value as the ellipse deforms into a circle while the kinetic energy reaches its peak value at the same instant. The total energy is nearly conserved (826.73 ± 7.03).

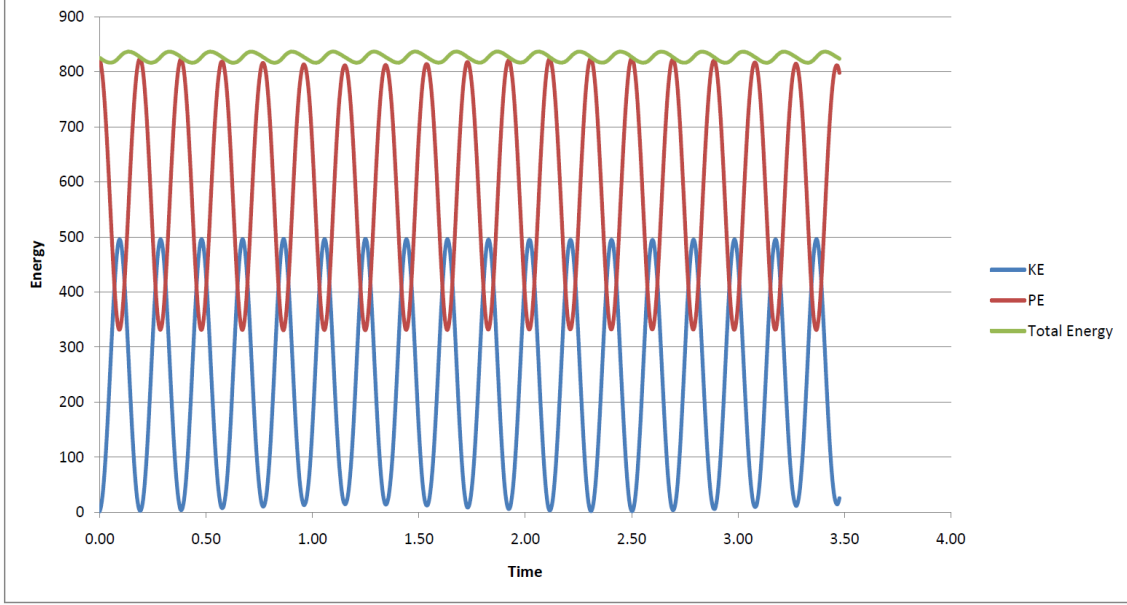


Figure 1: Energy of an ellipse

Next, we look at the order of accuracy in time for our integrator. We run the same undamped ellipse simulation at varying timesteps (0.005, 0.0025, 0.00125). Since our integrator nearly preserves energy and we do not have an analytic solution for our shape, we compare the total energy over time corresponding to each of these timesteps (see Figure 2). We notice that as we increase our timestep, there is a larger variance in the energy while the mean remains nearly the same. Further the energy profile becomes noisier at extrema for larger timesteps. By computing the standard deviation for these timesteps, we find that there is a linear relation between timestep and the standard deviation of the total energy of the curve, and hence a quadratic relationship for the variance (as seen in the graph). This is to be expected since we derived a first order variational time integrator. Spatial resolution does not appear to affect the total energy of the system.

Finally, we illustrate the evolution of some curves using our equations of dynamics (see Figure 3).

While our method produces visually plausible results, there are certain limitations of our current formulation. The modified potential energy function (sum of squares of lengths) does not exactly match the traditional surface tension formulation which instead uses length. This might make it difficult to compare results with existing work in the graphics and physics literature. Further, while we conserve mass, we do not preserve uniform density all over the loop but instead preserve it only

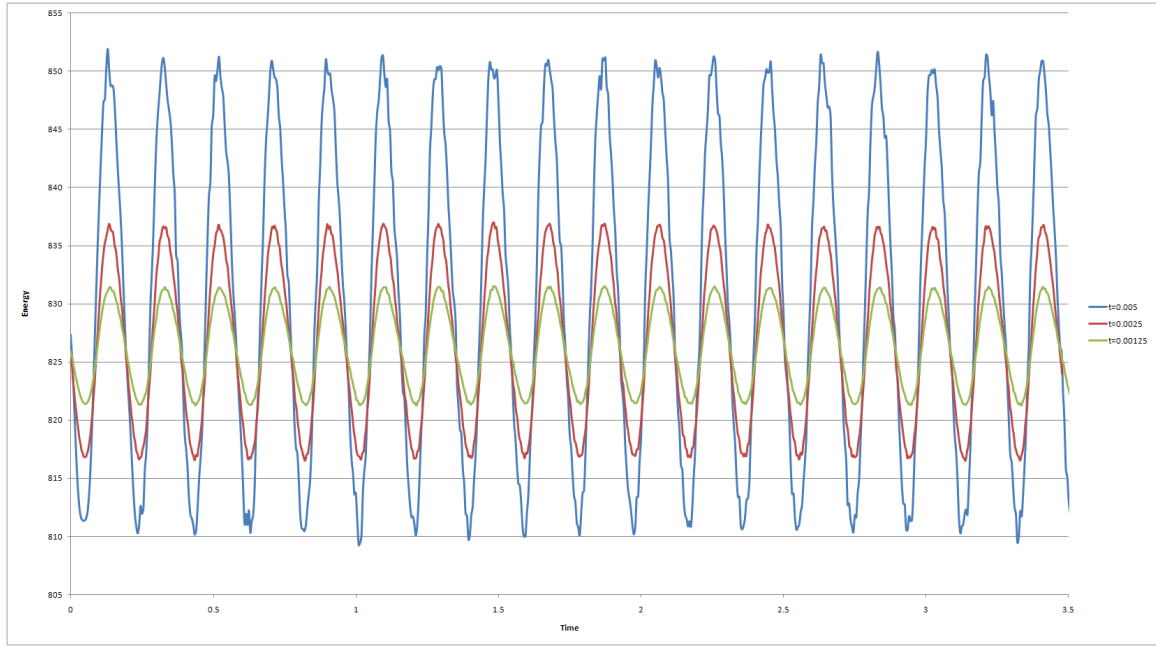


Figure 2: Energy profiles for different timesteps

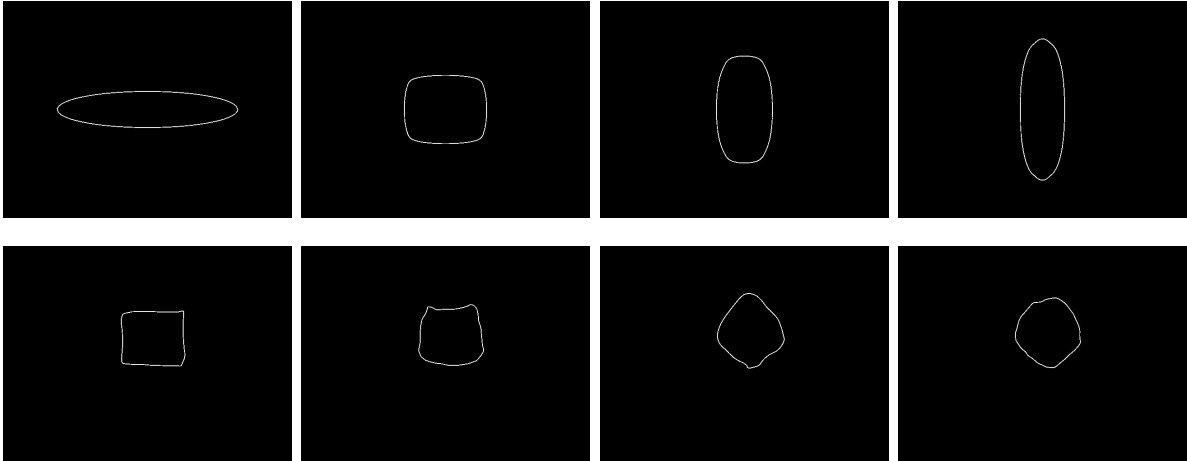


Figure 3: Evolution of various curves in time

along each edge. We assume that the mass of the bubble is purely on the boundary rather than over the area of the shape, similar to a soap bubble. Water drops would have to be treated differently. It would also be interesting to apply our method to a manifold mesh in 3D and handle topological changes due to self collisions. We would like to explore this in our future work.

A Vector product variant of integration by parts

Suppose we want to integrate the following vector product expression by parts to move the derivative on \mathbf{f} from \mathbf{v} (each of dimension n):

$$\int_0^{x_0} \mathbf{f}(x)^T \mathbf{v}'(x) dx \quad (44)$$

Then we expand the dot product into separate components and re-write the equation as:

$$\int_0^{x_0} f_1(x) v_1'(x) dx + \dots + \int_0^{x_0} f_n(x) v_n'(x) dx \quad (45)$$

Integrating each expression by parts gives:

$$= f_1(x) v_1(x) \Big|_0^{x_0} - \int_0^{x_0} f_1'(x) v_1(x) dx + \dots \quad (46)$$

$$= \mathbf{f}(x)^T \mathbf{v}(x) \Big|_0^{x_0} - \int_0^{x_0} \mathbf{f}'(x)^T \mathbf{v}(x) dx \quad (47)$$

If \mathbf{v} is zero at the boundary, we get:

$$\int_0^{x_0} \mathbf{f}(x)^T \mathbf{v}'(x) dx = - \int_0^{x_0} \mathbf{f}'(x)^T \mathbf{v}(x) dx \quad (48)$$

B Vector product variant of the fundamental lemma of calculus of variations

We want to prove the following, when $\mathbf{v}(x)$ is an arbitrary test function:

$$\int_0^{x_0} \mathbf{f}(x)^T \mathbf{v}(x) dx = 0 \Rightarrow \mathbf{f}(x) = 0 \quad (49)$$

We split the vector dot product into different components and rewrite the equation as:

$$\int_0^{x_0} f_1(x) v_1(x) dx + \dots + \int_0^{x_0} f_n(x) v_n(x) dx = 0 \quad (50)$$

Now suppose the function $\mathbf{f} = (f_1(x), \dots, f_n(x))^T \neq \mathbf{0}$, i.e. some components are non zero. Also without loss of generality assume that they are non zero for infinitesimally small intervals. Treating each component separately, we can always construct a test function such that the integral for each component becomes strictly positive (similar argument to the fundamental lemma for scalars). Therefore the sum in Equation (50) will be strictly greater than 0. This a contradiction because we assumed the equation to be true for all test function \mathbf{v} . Therefore, the result in Equation (49) is true since all the components of \mathbf{f} have to be uniformly zero.

References

STERN, A., AND DESBRUN, M. 2006. Discrete geometric mechanics for variational time integrators. In *SIGGRAPH '06: ACM SIGGRAPH 2006 Courses*, ACM, New York, NY, USA, 75–80.